J Math Chem (2010) 48:653–675
DOI 10.1007/s10910-010-9698-y

ORIGINAL PAPER

# Asymptotic behaviors of a delay difference system of plankton allelopathy

**Wenjie Qin · Zhijun Liu**

Received: 11 December 2009 / Accepted: 7 May 2010 / Published online: 21 May 2010


**Abstract** The main purpose of this paper is to study a general delay difference system of the growth of two species of plankton having competing and allelopathic effects on each other. We first show that the system is permanent under some suitable assumptions. Next, by using the continuous theorem of coincidence degree theory and constructing a Lyapunov function, a set of sufficient conditions which guarantee the existence and global attractivity of positive periodic solutions are obtained. Finally, two examples together with their numerical simulations are presented to illustrate the feasibility of our main results.

**Keywords** Delay difference system · Permanence · Global attractivity · Periodic solutions · Allelopathy

## 1 Introduction

Allelopathy is a biological phenomenon that is characteristic of any process involving secondary metabolites produced by some plant, algae, bacteria and fungi that influences the growth and development of biological systems, which was first used by Hans [1] in 1937 to describe biochemical interactions that inhibit the growth of neighboring plants, by another plant. This phenomenon exists in many fields such as agriculture, desert shrubs, forests and so on. Especially, the dynamic behaviors of phytoplankton are of great interest in mathematical biology, which deeply attract the attention of many scholars. Therefore, the study of tremendous fluctuations in abundance of many phytoplankton communities is one of the popular subjects in aquatic ecology. Recently, many investigations have been conducted to study the effect of toxic

W. Qin · Z. Liu (✉)
Department of Mathematics, Hubei University for Nationalities,
445000 Enshi, People's Republic of China
e-mail: zhijun_liu47@hotmail.com; zjliu1008@gmail.com

substances on plankton allelopathy by using mathematical models and many excellent results have been derived (see [2–13] and references cited therein). Maynard Smith [14] and finally Chattopadhyay [15] modified a traditional Lotka-Volterra two species competition system by considering that each species produces a substance toxic to the other but only when the other is present and studied the stability properties of the following system

$$\begin{cases} \frac{dx_1(t)}{dt} = x_1(t)[r_1 - a_{11}x_1(t) - a_{12}x_2(t) - b_1x_1(t)x_2(t)], \\ \frac{dx_2(t)}{dt} = x_2(t)[r_2 - a_{21}x_1(t) - a_{22}x_2(t) - b_2x_1(t)x_2(t)], \end{cases} \tag{1.1}$$

where $x_1(t), x_2(t)$ stand for the population densities of two competing species; $r_1, r_2$ are intrinsic growth rates of two competing species; $a_{11}, a_{22}$ are the coefficients of intra-specific competition of the first and second species, respectively; $a_{12}, a_{21}$ are the coefficients of inter-specific competition of the first and second species, respectively; $r_1/a_{11}$ and $r_2/a_{22}$ are environmental carrying capacities. The terms $b_1x_1(t)x_2(t)$ and $b_2x_1(t)x_2(t)$ denote the effect of toxic substances. Here, they made the assumption that each species produces a substance toxic to the other, only when the other is present.

Noticing that the production of allelopathic substance by the competing species will not be instantaneous, but mediated by some time lag required for maturity of the species. On the other hand, considering any biological and environmental parameters are naturally subject to fluctuation in time. The effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Therefore, Motivated by Jin and Ma [16], considering the distributed delay and periodic changing of environment, we have modified system (1.1) in the form

$$\begin{cases} \frac{dx_1(t)}{dt} = x_1(t)[r_1(t) - a_{11}(t)x_1(t) - a_{12}(t)x_2(t) - b_1(t)x_1(t) \int_{-\tau}^{0} K_2(s)x_2(t+s)ds], \\ \frac{dx_2(t)}{dt} = x_2(t)[r_2(t) - a_{21}(t)x_1(t) - a_{22}(t)x_2(t) - b_2(t)x_2(t) \int_{-\tau}^{0} K_1(s)x_1(t+s)ds], \end{cases} \tag{1.2}$$

where $r_i(t), a_{ij}(t), b_i(t), i, j = 1, 2$ are nonnegative continuous functions, $\tau$ is a positive constant, $K_i(t) \in C([-\tau, 0), (0, +\infty))$ and $\int_{-\tau}^{0} K_i(s)ds = 1, i = 1, 2$.

Although much progress has been seen in the model of plankton allelopathy, such models are not well studied in the sense that most results are continuous time versions related. Many authors [17–19]have argued that the discrete time models governed by difference equations are more appropriate than the continuous ones when populations have a short life expectancy, non-overlapping generations in the real word. Discrete time models can also provide efficient computational models of continuous models for numerical simulations. So it is reasonable to study discrete time competitive systems governed by difference equations. Though there are many excellent works about the system of plankton allelopathy with discrete time delay [5,13], no such work has been done for the (1.2), which has a distributed delay.

A semi-discretization technique and method in formulating a discrete time analogue of the continuous time system (1.2) is proposed. This method has been employed elsewhere(see details in [20–22]) in the formulation of discrete-time analogues of continuous-time dynamical systems modelling population dynamics. For the simplicity and convenience of exposition, throughout this paper, we let $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{R}^+$ and $\mathbb{R}^2$ denote the sets of all integers, nonnegative integers, nonnegative real numbers, and two-dimensional Euclidian vector space, respectively. We begin approximating the continuous time system by replacing the integral term with discrete sums of the form

$$\begin{aligned}\int_{-\tau}^{0} K_i(s)x_i(t+s)\mathrm{d}s &= \int_{0}^{\tau} K_i(-s)x_i(t-s)\mathrm{d}s \\ &\approx \sum_{\left[\frac{s}{h}\right]=0}^{M} K_i\left(-\left[\frac{s}{h}\right]h\right) x_i\left(\left[\frac{t}{h}\right]h-\left[\frac{s}{h}\right]h\right) w(h)\end{aligned}$$

for $t \in [nh, (n+1)h]$, $s \in [ph, (p+1)h]$, $n, p \in \mathbb{Z}^+$, $i = 1, 2$, where $[\cdot]$ denotes the greatest integer function. $M = \left[\frac{\tau}{h}\right]$, $w(t) = h + O(h)$ for $h > 0$, and $h$ is a fixed number denoting a uniform discretization step size. Following this, we approximate (1.2) by differential equations with piecewise constant arguments of the form

$$\begin{cases} \frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = x_1(t)\Bigg[r_1\left(\left[\frac{t}{h}\right]h\right) - a_{11}\left(\left[\frac{t}{h}\right]h\right)x_1\left(\left[\frac{t}{h}\right]h\right) - a_{12}\left(\left[\frac{t}{h}\right]h\right)x_2\left(\left[\frac{t}{h}\right]h\right) \\ \qquad - b_1\left(\left[\frac{t}{h}\right]h\right)x_1\left(\left[\frac{t}{h}\right]h\right)\sum\limits_{\left[\frac{s}{h}\right]=0}^{M} K_2\left(-\left[\frac{s}{h}\right]h\right)x_2\left(\left[\frac{t}{h}\right]h-\left[\frac{s}{h}\right]h\right)h\Bigg], \\ \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = x_2(t)\Bigg[r_2\left(\left[\frac{t}{h}\right]h\right) - a_{21}\left(\left[\frac{t}{h}\right]h\right)x_1\left(\left[\frac{t}{h}\right]h\right) - a_{22}\left(\left[\frac{t}{h}\right]h\right)x_2\left(\left[\frac{t}{h}\right]h\right) \\ \qquad - b_2\left(\left[\frac{t}{h}\right]h\right)x_2\left(\left[\frac{t}{h}\right]h\right)\sum\limits_{\left[\frac{s}{h}\right]=0}^{M} K_1\left(-\left[\frac{s}{h}\right]h\right)x_1\left(\left[\frac{t}{h}\right]h-\left[\frac{s}{h}\right]h\right)h\Bigg] \end{cases} \tag{1.3}$$

for $t \in [nh, (n+1)h)$, $s \in [ph, (p+1)h)$, $n, p \in \mathbb{Z}^+$. Noting that $\left[\frac{t}{h}\right] = n$, $\left[\frac{s}{h}\right] = p$, we rewrite (1.3) as

$$\begin{cases} \frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = x_1(t)\Bigg[r_1(nh) - a_{11}(nh)x_1(nh) - a_{12}(nh)x_2(nh) \\ \qquad -b_1(nh)x_1(nh)\sum\limits_{p=0}^{M} k_2(ph)x_2(nh-ph)h\Bigg], \\ \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = x_2(t)\Bigg[r_2(nh) - a_{21}(nh)x_1(nh) - a_{22}(nh)x_2(nh) \\ \qquad -b_2(nh)x_2(nh)\sum\limits_{p=0}^{M} k_1(ph)x_1(nh-ph)h\Bigg], \end{cases} \tag{1.4}$$

where $k_i(ph) = K_i(-ph), i = 1, 2$. We integrate system (1.4) over $[nh, t)$, where $t < (n+1)h$, then

$$\begin{cases} x_1(t) = x_1(nh)\exp\Bigg\{\Bigg[r_1(nh) - a_{11}(nh)x_1(nh) - a_{12}(nh)x_2(nh) \\ \qquad\qquad -b_1(nh)x_1(nh)\sum_{p=0}^{M} k_2(ph)x_2(nh-ph)h\Bigg](t-nh)\Bigg\}, \\ x_2(t) = x_2(nh)\exp\Bigg\{\Bigg[r_2(nh) - a_{21}(nh)x_1(nh) - a_{22}(nh)x_2(nh) \\ \qquad\qquad -b_2(nh)x_2(nh)\sum_{p=0}^{M} k_1(ph)x_1(nh-ph)h\Bigg](t-nh)\Bigg\}. \end{cases} \tag{1.5}$$

Denoting $x_i(n) = x_i(nh)$, $r_i(n) = r_i(nh)h$, $a_{ij}(n) = a_{ij}(nh)h$, $b_i(n) = b_i(nh)h$ and $k_i(n) = k_i(nh)h$, $i, j = 1, 2$. Meanwhile, allowing $t \to (n+1)h$ in (1.5) and simplifying, we obtain a discrete time analogue of the continuous time system (1.2) given by

$$\begin{cases} x_1(n+1) \;\; = x_1(n)\exp\Bigg\{r_1(n) - a_{11}(n)x_1(n) - a_{12}(n)x_2(n) \\ \qquad\qquad -b_1(n)x_1(n)\sum_{p=0}^{M} k_2(p)x_2(n-p)\Bigg\}, \\ x_2(n+1) \;\; = x_2(n)\exp\Bigg\{r_2(n) - a_{21}(n)x_1(n) - a_{22}(n)x_2(n) \\ \qquad\qquad -b_2(n)x_2(n)\sum_{p=0}^{M} k_1(p)x_1(n-p)\Bigg\}, \\ x_i(\Phi) \geq 0, \quad \Phi \in [-p, 0] \cap \mathbb{Z}; \quad x_i(0) > 0, \quad i = 1, 2. \end{cases} \tag{1.6}$$

One can show that the discrete-time analogue (1.6) converges to continuous-time system (1.2). Here, as far as system (1.6) is concerned, several issues are proposed.

(1) Is it possible to obtain a set of sufficient conditions which guarantee the permanence of the system?
(2) Is it possible to obtain a set of sufficient conditions which guarantee the existence of positive periodic solutions of the system? If exists, whether it is globally attractive or not.

The aim of this paper is to give an affirmative answer to the above issues. To the best of our knowledge, no work have been done for the delay difference system (1.6).

Throughout this paper, it is assumed that $\{r_i(n)\}$, $\{a_{ij}(n)\}$, $\{b_i(n)\}$, $i, j = 1, 2$ are bounded nonnegative sequences, $K_i(p), i = 1, 2$ is bounded positive sequence, we use the notation $f^u = \sup_{t\in\mathbb{Z}^+} f(t)$, $f^l = \inf_{t\in\mathbb{Z}^+} f(t)$ for any bounded sequence $\{f(t)\}$. Meanwhile, we make a convention that $\sum_{t=m}^{n} f(t) = 0$ if $m > n$.

The rest of this paper is organized as follows. We study the persistent property of (1.6) in Sect. 2 and the periodic property in Sect. 3. As an application of our main results, we give two examples in Sect. 4.

## 2 Permanence

This section devotes to investigate the persistent property of system (1.6). To do so, we need to make some preparations. For convenience, we first introduce the definition of permanence.

**Definition 2.1** System (1.6) is said to be permanent if there exist positive constants $m_i$ and $M_i$ such that

$$m_i \leq \liminf_{n\to+\infty} x_i(n) \leq \limsup_{n\to+\infty} x_i(n) \leq M_i, \quad i = 1, 2$$

for any positive solution $\{(x_1(n), x_2(n))^\top\}$ of (1.6).

Next, we will introduce the following Lemmas 2.1 and 2.2 which are given by Lemmas 1 and 2 in [23], respectively.

**Lemma 2.1** *Assume that* $\{x(n)\}$ *satisfies* $x(n) > 0$ *and*

$$x(n+1) \leq x(n)\exp\Big[r(n)(1 - \alpha x(n))\Big]$$

*for* $n \in [n_1, +\infty)$*, where* $\alpha$ *is a positive constant and* $n_1 \in \mathbb{Z}^+$*. Then*

$$\limsup_{n\to+\infty} x(n) \leq \frac{1}{\alpha r^u}\exp(r^u - 1).$$

**Lemma 2.2** *Assume that* $\{x(n)\}$ *satisfies* $x(n) > 0$ *and*

$$x(n+1) \geq x(n)\exp\Big[r(n)(1 - \alpha x(n))\Big]$$

*for* $n \in [n_2, +\infty)$*,* $\limsup_{n\to+\infty} x(n) \leq M$ *and* $x(n_2) > 0$*, where* $\alpha$ *is a positive constant such that* $\alpha M > 1$ *and* $n_2 \in \mathbb{Z}^+$*. Then*

$$\liminf_{n\to+\infty} x(n) \geq \frac{1}{\alpha}\exp\Big[r^u(1 - \alpha M)\Big].$$

Before stating Theorem 2.1, for the simplicity and convenience of exposition, we set

$$\Delta_i^\varepsilon = \frac{a_{ii}^u + b_i^u(M_j+\varepsilon)(M+1)k_j^u}{r_i^l - a_{ij}^u(M_j+\varepsilon)}, \quad \Delta_i = \lim_{\varepsilon\to 0}\Delta_i^\varepsilon, \qquad i, j = 1, 2 \text{ and } i \neq j.$$

Here, $\varepsilon$ is a sufficiently small positive constant. Meanwhile, $M_1$ and $M_2$ are defined in (2.3) and (2.4), respectively.

We are now in a position to state our main result of this section on the permanence of (1.6).

**Theorem 2.1** *Assume that*

$$\min\Big\{r_1^l - a_{12}^u M_2,\ \ r_2^l - a_{21}^u M_1\Big\} > 0, \tag{2.1}$$

$$\min\Big\{\Delta_1 M_1,\ \ \Delta_2 M_2\Big\} > 1. \tag{2.2}$$

*Then system* (1.6) *is permanent.*

*Proof* Since $x_1(0) > 0$ and $x_2(0) > 0$, then we let $\{(x_1(n), x_2(n))^\top\}$ be any positive solution of (1.6). To prove Theorem 2.1, we divide into the following two steps.

*Step I.* According to Definition 2.1, we will prove $\{(x_1(n), x_2(n))^\top\}$ satisfies

$$\limsup_{n\to+\infty} x_1(n) \le M_1,\ \ \limsup_{n\to+\infty} x_2(n) \le M_2.$$

It follows from the first equation of (1.6) that

$$x_1(n+1) \le x_1(n)\exp[r_1(n) - a_{11}^l x_1(n)] \le x_1(n)\exp\left\{r_1(n)\left[1 - \frac{a_{11}^l}{r_1^u}x_1(n)\right]\right\}.$$

Applying Lemma 2.1, we can obtain that

$$\limsup_{n\to+\infty} x_1(n) \le \frac{\exp(r_1^u - 1)}{a_{11}^l} \doteq M_1. \tag{2.3}$$

From the second equation of (1.6), similar to above analysis, we have

$$\limsup_{n\to+\infty} x_2(n) \le \frac{\exp(r_2^u - 1)}{a_{22}^l} \doteq M_2. \tag{2.4}$$

*Step II.* In this step we will prove $\{(x_1(n), x_2(n))^\top\}$ satisfies

$$\liminf_{n\to+\infty} x_1(n) \ge m_1,\ \ \liminf_{n\to+\infty} x_2(n) \ge m_2,$$

where $m_1, m_2$ are defined in (2.7) and (2.8), respectively.
For any sufficiently small $\varepsilon > 0$, it follows from (2.1) and (2.2) that

$$r_1^l - a_{12}^u(M_2 + \varepsilon) > 0,\quad \Delta_1^\varepsilon M_1 > 1. \tag{2.5}$$

For the above $\varepsilon > 0$, in view of (2.4), there exists a positive integer $n_0$ such that $x_2(n) \leq M_2 + \varepsilon$ for all $n \geq n_0$. Thus, by the first equation of (1.6), it gives that

$$\begin{aligned} x_1(n+1) &\geq x_1(n)\exp\Big\{r_1^l - a_{12}^u(M_2+\varepsilon) - [a_{11}^u + b_1^u(M_2+\varepsilon)(M+1)k_2^u]x_1(n)\Big\} \\ &= x_1(n)\exp\Big\{[r_1^l - a_{12}^u(M_2+\varepsilon)][1-\Delta_1^\varepsilon x_1(n)]\Big\}. \end{aligned} \tag{2.6}$$

By Lemma 2.2, it follows from (2.3), (2.5) and (2.6) that

$$\liminf_{n\to+\infty} x_1(n) \geq \frac{\exp\{[r_1^l - a_{12}^u(M_2+\varepsilon)](1-\Delta_1^\varepsilon M_1)\}}{\Delta_1^\varepsilon},$$

and letting $\varepsilon \to 0$, we can obtain that

$$\liminf_{n\to+\infty} x_1(n) \geq \frac{\exp[(r_1^l - a_{12}^u M_2)(1-\Delta_1 M_1)]}{\Delta_1} \doteq m_1. \tag{2.7}$$

Similar to above argument, we can also show that

$$\liminf_{n\to+\infty} x_2(n) \geq \frac{\exp[(r_2^l - a_{21}^u M_1)(1-\Delta_2 M_2)]}{\Delta_2} \doteq m_2 \tag{2.8}$$

from the second equation of (1.6).

Combing the above Steps I and II, we have proved that (1.6) is permanent. This completes the proof. □

## 3 Existence and global attractivity of positive periodic solutions

In this section, we investigate the periodic property of system (1.6). At first, by using the continuous theorem of coincidence degree theory, sufficient conditions for the existence of positive periodic solutions are derived.

For the convenience of the readers, we always assume that $r_i, a_{ij}, b_i, i, j = 1, 2$ are nonnegative sequences with common periodic $\omega$ for (1.6), where $\omega$ is a fixed positive integer, denotes the prescribed common period of the parameters in (1.6). The exponential form of the equations in (1.6) ensures that any forward trajectory $\{(x_1(n), x_2(n))^\top\}$ of (1.6) with $n > 0$ remains positive for all $n \in \mathbb{Z}^+$.

In order to obtain sufficient conditions for the existence of positive periodic solutions of (1.6), for convenience, we shall summarize in the following a few concepts and results from [24] that will be useful in this section.

Let $X$ and $Y$ be two Banach spaces. Consider an operator equation

$$Lx = \lambda Nx, \quad \lambda \in (0, 1),$$

where $L : \text{Dom } L \cap X \to Y$ is a linear operator and $\lambda$ is a parameter. Let $P$ and $Q$ denote two projectors such that

$$P : X \cap \text{Dom } L \to \text{Ker } L \text{ and } Q : Y \to Y/\text{Im } L.$$

Denote by $J : \text{Im } Q \to \text{Ker } L$ an isomorphism of Im $Q$ onto Ker $L$. Recall that a linear mapping $L : \text{Dom } L \cap X \to Y$ with Ker $L = L^{-1}(0)$ and Im $L = L(\text{Dom } L)$, will be called a Fredholm mapping if the following two conditions hold:

($S_1$) Ker $L$ has a finite dimension;
($S_2$) Im $L$ is closed and has a finite codimension.

Recall also that the codimension of Im $L$ is the dimension of $Y/\text{Im } L$, i.e., the dimension of the cokernel coker $L$ of $L$.

When $L$ is a Fredholm mapping, its index is the integer Ind $L$ = dim Ker $L$ − codim Im $L$.

We shall say that a mapping $N$ is $L-$compact on $\Omega$ if the mapping $QN : \bar{\Omega} \to Y$ is continuous. $QN(\bar{\Omega})$ is bounded, and $K_p(I-Q)N : \bar{\Omega} \to X$ is compact, i.e., it is continuous and $K_p(I-Q)N(\bar{\Omega})$ is relatively compact, where $K_p : \text{Im } L - \text{Dom } L \cap \text{Ker } P$ is an inverse of the restriction $L_p$ of $L$ to Dom $L \cap$ Ker $P$, so that $LK_p = I$ and $K_p = I - P$. In the sequel, we will use the following result of Mawhin [24].

**Lemma 3.1** *Let X and Y be two Banach spaces and let L be a Fredholm mapping of index zero. Assume that $N : \bar{\Omega} \to Y$ is $L-$compact on $\bar{\Omega}$ with $\Omega$ open bounded in X. Furthermore assume:*

($D_1$) *for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap$ Dom $L$, $Lx \neq \lambda Nx$;*
($D_2$) *$QNx \neq 0$ for each $x \in \partial\Omega \cap$ Ker $L$;*
($D_3$) *$deg\{JQNx, \Omega \cap \text{Ker } L, 0\} \neq 0$.*

*Then the equation $Lx = Nx$ has at least one solution in Dom $L \cap \bar{\Omega}$.*

In what follows, we shall use the following notations:

$$I_\omega = \{0, 1, \ldots, \omega - 1\}, \quad \bar{g} = \frac{1}{\omega}\sum_{t=0}^{\omega-1} g(t), \quad g^L = \min_{t\in I_\omega} g(t), \quad g^U = \max_{t\in I_\omega} g(t),$$

where $\{g(t)\}$ is an $\omega-$periodic sequence of real numbers defined for $t \in \mathbb{Z}$. The following result was given by Lemma 3.2 in [25].

**Lemma 3.2** *Let $f : \mathbb{Z} \to \mathbb{R}$ be $\omega-$periodic, that is, $f(n+\omega) = f(n)$, then for any fixed $n_1, n_2 \in I_\omega$ and any $n \in \mathbb{Z}$, one has*

$$f(n) \leq f(n_1) + \sum_{s=0}^{\omega-1} |f(s+1) - f(s)|,$$

$$f(n) \geq f(n_2) - \sum_{s=0}^{\omega-1} |f(s+1) - f(s)|.$$

Denote

$$l_2 = \Big\{x = \{x(n)\} : x(n) \in \mathbb{R}^2, n \in \mathbb{Z}\Big\}.$$

For $a = (a_1, a_2)^\top \in \mathbb{R}^2$, define $|a| = \max\{a_1, a_2\}$. Let $l^\omega \subset l_2$ denote the subspace of all $\omega-$periodic sequences equipped with the usual supremum norm $\|\cdot\|$, i.e.,

$$\|x\| = \max_{n\in I_\omega} |x(n)| \ \text{ for } \ x = \{x(n) : n \in \mathbb{Z}\} \in l^\omega.$$

Then it follows that $l^\omega$ is a finite dimensional Banach space.

Let

$$l_0^\omega = \Big\{x = \{x(n) \in l^\omega\} : \sum_{n=0}^{\omega-1} x(n) = 0\Big\},$$
$$l_c^\omega = \Big\{x = \{x(n) \in l^\omega\} : x(n) = h \in \mathbb{R}^2, n \in \mathbb{Z}\Big\}.$$

Then it follows that $l_0^\omega$ and $l_c^\omega$ are both closed linear subspaces of $l^\omega$ and

$$l^\omega = l_0^\omega \oplus l_c^\omega, \ \ \dim l_c^\omega = 2.$$

We are now in a position to state one main result of this section on the existence of positive periodic solutions of (1.6).

**Theorem 3.1** *Assume that*

$$\frac{\bar{a}_{ii}}{\bar{a}_{ji}} \geq \min\left\{\frac{\bar{b}_i \sum_{p=0}^{M} k_j(p)}{\bar{b}_j \sum_{p=0}^{M} k_i(p)}, \frac{\bar{r}_i}{\bar{r}_j} e^{2\bar{r}_i\omega}\right\}, \quad i, j = 1, 2 \ \textit{and} \ i \neq j. \tag{3.1}$$

*Then system (1.6) has at least one positive $\omega-$periodic solution.*

*Proof* We first make the change of variables

$$y_1(n) = \ln\{x_1(n)\}, \ \ y_2(n) = \ln\{x_2(n)\}.$$

Then (1.6) can be reformulated as

$$\begin{cases} y_1(n+1) - y_1(n) = r_1(n) - a_{11}(n)e^{y_1(n)} - a_{12}(n)e^{y_2(n)} \\ \qquad\qquad\qquad\qquad -b_1(n)e^{y_1(n)} \sum\limits_{p=0}^{M} k_2(p)e^{y_2(n-p)}, \\ y_2(n+1) - y_2(n) = r_2(n) - a_{21}(n)e^{y_1(n)} - a_{22}(n)e^{y_2(n)} \\ \qquad\qquad\qquad\qquad -b_2(n)e^{y_2(n)} \sum\limits_{p=0}^{M} k_1(p)e^{y_1(n-p)}. \end{cases} \tag{3.2}$$

It is easy to see that if (3.2) has one $\omega-$periodic solution, then (1.6) has one positive $\omega-$periodic solution. Therefore, to complete the proof, it is only to show that (3.2) has at least one $\omega-$periodic solution.

Set $X = Y = l^\omega$. Denote by $L : X \to X$ the difference operator given by $Ly = \{(Ly)(n)\}$ with

$$(Ly)(n) = y(n+1) - y(n) \quad \text{for} \quad y \in X \quad \text{and} \quad n \in \mathbb{Z},$$

and $N : X \to X$ as follows

$$N\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} r_1(n) - a_{11}(n)e^{y_1(n)} - a_{12}(n)e^{y_2(n)} - b_1(n)e^{y_1(n)} \sum\limits_{p=0}^{M} k_2(p)e^{y_2(n-p)} \\ r_2(n) - a_{21}(n)e^{y_1(n)} - a_{22}(n)e^{y_2(n)} - b_2(n)\mathrm{e}^{y_2(n)} \sum\limits_{p=0}^{M} k_1(p)e^{y_1(n-p)} \end{bmatrix}$$

for any $(y_1, y_2)^\top \in X$ and $n \in \mathbb{Z}$. It is easy to see that $L$ is a bounded linear operator and

$$\text{Ker } L = l_c^\omega, \quad \text{Im } L = l_0^\omega, \quad \dim \text{Ker } L = 2 = \text{codim Im } L,$$

then we get that $L$ is a Fredholm mapping of index zero.

Define

$$P\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Q\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega} \sum\limits_{s=0}^{\omega-1} y_1(s) \\ \frac{1}{\omega} \sum\limits_{s=0}^{\omega-1} y_2(s) \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in X = Y.$$

It is not difficult to show that $P$ and $Q$ are continuous projectors such that

$$\text{Im } P = \text{Ker } L, \quad \text{Ker } Q = \text{Im } L = \text{Im } (I - Q).$$

Furthermore, the inverse (to $L$) $K_p : \text{Im } L \to \text{Dom } L \cap \text{Ker } P$ exists and is given by

$$K_p(y) = \sum_{s=0}^{n-1} y(s) - \frac{1}{\omega} \sum_{s=0}^{\omega-1} (\omega - s) y(s).$$

Then $QN : X \to Y$ and $K_p(I - Q)N : X \to X$ are given by

$$QNy = \frac{1}{\omega} \sum_{s=0}^{\omega-1} Ny(s)$$

and

$$K_p(I-Q)Ny=\sum_{s=0}^{n-1}Ny(s)-\frac{1}{\omega}\sum_{s=0}^{\omega-1}(\omega-s)Ny(s)-\left(\frac{n}{\omega}-\frac{1+\omega}{2\omega}\right)\sum_{s=0}^{\omega-1}Ny(s).$$

In order to apply Arzela-Ascoli theorem, we need to search for an appropriately open, bounded subset $\Omega$.

Corresponding to the operator equation $Ly=\lambda Ny$, $\lambda\in(0,1)$, we have

$$\begin{cases} y_1(n+1)-y_1(n)=\lambda\Bigg[r_1(n)-a_{11}(n)e^{y_1(n)}-a_{12}(n)e^{y_2(n)} \\ \qquad\qquad -b_1(n)e^{y_1(n)}\sum\limits_{p=0}^{M}k_2(p)e^{y_2(n-p)}\Bigg], \\ y_2(n+1)-y_2(n)=\lambda\Bigg[r_2(n)-a_{21}(n)e^{y_1(n)}-a_{22}(n)e^{y_2(n)} \\ \qquad\qquad -b_2(n)\mathrm{e}^{y_2(n)}\sum\limits_{p=0}^{M}k_1(p)e^{y_1(n-p)}\Bigg]. \end{cases} \tag{3.3}$$

Suppose that $y=\{y(n)\}=\left\{\left(y_1(n),y_2(n)\right)^{\top}\right\}\in X$ is a solution of (3.3) for a certain $\lambda\in(0,1)$. Summing both sides of (3.3) from 0 to $\omega-1$ with respect to $n$, we can derive

$$\begin{cases} \bar{r}_1\omega=\sum\limits_{n=0}^{\omega-1}\left[a_{11}(n)e^{y_1(n)}+a_{12}(n)e^{y_2(n)}+b_1(n)e^{y_1(n)}\sum\limits_{p=0}^{M}k_2(p)e^{y_2(n-p)}\right], \\ \bar{r}_2\omega=\sum\limits_{n=0}^{\omega-1}\left[a_{21}(n)e^{y_1(n)}+a_{22}(n)e^{y_2(n)}+b_2(n)e^{y_2(n)}\sum\limits_{p=0}^{M}k_1(p)\mathrm{e}^{y_1(n-p)}\right]. \end{cases} \tag{3.4}$$

It follows from (3.3) and (3.4) for $i,j=1,2$ and $i\neq j$ that

$$\begin{aligned} \sum_{n=0}^{\omega-1}|y_i(n+1)-y_i(n)| &\le \lambda\Bigg\{\sum_{n=0}^{\omega-1}\Bigg[r_i(n)+a_{ii}(n)e^{y_i(n)}+a_{ij}(n)e^{y_j(n)} \\ &\qquad +b_i(n)e^{y_i(n)}\sum_{p=0}^{M}k_j(p)e^{y_j(n-p)}\Bigg]\Bigg\} \\ &\le 2\bar{r}_i\omega. \end{aligned} \tag{3.5}$$

Since $y=\{y(n)\}\in X$, there exist $\xi_i\in I_\omega$ such that

$$y_i(\xi_i)=\min_{n\in I_\omega}\{y_i(n)\},\quad i=1,2.$$

It follows from (3.4) that

$$\begin{aligned}\bar{r}_i\omega &\geq \sum_{n=0}^{\omega-1}\left[a_{ii}(n)e^{y_i(\xi_i)}+a_{ij}(n)e^{y_j(\xi_j)}+b_i(n)e^{y_i(\xi_i)}\sum_{p=0}^{M}k_j(p)e^{y_j(\xi_j)}\right]\\ &\geq \sum_{n=0}^{\omega-1}a_{ii}(n)e^{y_i(\xi_i)}=\omega\bar{a}_{ii}e^{y_i(\xi_i)},\quad i,j=1,2 \text{ and } i\neq j,\end{aligned}$$

which implies

$$y_i(\xi_i)\leq \ln\frac{\bar{r}_i}{\bar{a}_{ii}}=\ln A_i, \tag{3.6}$$

where $A_i=\frac{\bar{r}_i}{\bar{a}_{ii}}$, $i=1,2$. By (3.5), (3.6) and Lemma 3.2, we have

$$y_i(n)\leq y_i(\xi_i)+\sum_{s=0}^{\omega-1}|y_i(s+1)-y_i(s)|\leq \ln A_i+2\bar{r}_i\omega,\quad i=1,2. \tag{3.7}$$

On the other hand, there also exist $\eta_i\in I_\omega$ such that

$$y_i(\eta_i)=\max_{n\in I_\omega}\{y_i(n)\},\quad i=1,2.$$

In view of (3.4), we can obtain

$$\begin{aligned}\bar{r}_i\omega &\leq \sum_{n=0}^{\omega-1}\left\{a_{ii}(n)e^{y_i(\eta_i)}+a_{ij}(n)e^{y_j(\eta_j)}+b_i(n)e^{y_i(\eta_i)}\sum_{p=0}^{M}k_j(p)e^{y_j(\eta_j)}\right\}\\ &=\left\{\left[\bar{a}_{ii}+\bar{b}_i\sum_{p=0}^{M}k_j(p)e^{y_j(\eta_j)}\right]e^{y_i(\eta_i)}+\bar{a}_{ij}e^{y_j(\eta_j)}\right\}\omega,\ \ i,j=1,2 \text{ and } i\neq j.\end{aligned} \tag{3.8}$$

Therefore,

$$\begin{aligned}\bar{r}_i\leq \left[\bar{a}_{ii}+\bar{b}_i\sum_{p=0}^{M}k_j(p)e^{y_j(\eta_j)}\right]e^{y_i(\eta_i)}+\bar{a}_{ij}e^{y_j(\eta_j)},\\ i,j=1,2 \quad\text{and}\quad i\neq j.\end{aligned} \tag{3.9}$$

Then

$$e^{y_i(\eta_i)} \geq \frac{\bar{r}_i - \bar{a}_{ij} e^{y_j(\eta_j)}}{\bar{a}_{ii} + \bar{b}_i \sum\limits_{p=0}^{M} k_j(p) e^{y_j(\eta_j)}} \geq \frac{\bar{r}_i - \bar{a}_{ij} \frac{\bar{r}_j}{\bar{a}_{jj}} e^{2\bar{r}_j \omega}}{\bar{a}_{ii} + \bar{b}_i \sum\limits_{p=0}^{M} k_j(p) \frac{\bar{r}_j}{\bar{a}_{jj}} e^{2\bar{r}_j \omega}}$$

$$= \frac{\bar{r}_i \bar{a}_{jj} - \bar{a}_{ij} \bar{r}_j e^{2\bar{r}_j \omega}}{\bar{a}_{ii} \bar{a}_{jj} + \bar{b}_i \sum\limits_{p=0}^{M} k_j(p) \bar{r}_j e^{2\bar{r}_j \omega}} \doteq B_i$$

for $i, j = 1, 2$ and $i \neq j$. That is,

$$y_i(\eta_i) \geq \ln B_i, \qquad i = 1, 2. \tag{3.10}$$

By (3.5), (3.10) and Lemma 3.2, we have

$$y_i(n) \geq y_i(\eta_i) - \sum_{s=0}^{\omega-1} |y_i(s+1) - y_i(s)| \geq \ln B_i - 2\bar{r}_i \omega, \qquad i = 1, 2. \tag{3.11}$$

Eqs. (3.7) and (3.11) imply

$$|y_i(n)| \leq \max\Big\{|\ln A_i + 2\bar{r}_i \omega|, |\ln B_i - 2\bar{r}_i \omega|\Big\} \doteq H_i, \qquad i = 1, 2. \tag{3.12}$$

Obviously, $A_i, B_i, H_i, i = 1, 2$ in (3.12) are independent of $\lambda$, respectively. Denote $H = H_1 + H_2 + h_0$, where $h_0$ is taken sufficiently large such that the unique solution $(y_1^*, y_2^*)^\top$ of the system of algebraic equation

$$\begin{cases} \bar{r}_1 = \bar{a}_{11} e^{y_1} + \bar{a}_{12} e^{y_2} + \bar{b}_1 e^{y_1} e^{y_2} \sum\limits_{p=0}^{M} k_2(p), \\ \bar{r}_2 = \bar{a}_{21} e^{y_1} + \bar{a}_{22} e^{y_2} + \bar{b}_2 e^{y_1} e^{y_2} \sum\limits_{p=0}^{M} k_1(p) \end{cases} \tag{3.13}$$

satisfies $\|(y_1^*, y_2^*)^\top\| = \max\{|y_1^*|, |y_2^*|\} < h_0$ (if system (3.13) has at least one solution). We let $\Omega := \{y : (y_1, y_2) \in X \Big| ||y|| < H\}$, thus the condition $(D_1)$ in Lemma 3.1 holds. When $y \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap \mathbb{R}^2,\ y = \{(y_1, y_2)^\top\}$, and $(y_1, y_2)^\top$ is a constant vector in $\mathbb{R}^2$ with $\|y\| = H$. If (3.13) has at least one solution, then

$$QN\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \bar{r}_1 - \bar{a}_{11} e^{y_1} - \bar{a}_{12} e^{y_2} - \bar{b}_1 e^{y_1} e^{y_2} \sum\limits_{p=0}^{M} k_2(p) \\ \bar{r}_2 - \bar{a}_{21} \mathrm{e}^{y_1} - \bar{a}_{22} e^{y_2} - \bar{b}_2 e^{y_1} e^{y_2} \sum\limits_{p=0}^{M} k_1(p) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If (3.13) does not have one solution, then it is obvious that

$$QN\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This implies that the condition $(D_2)$ in Lemma 3.1 holds.

Now we prove that the condition $(D_3)$ in Lemma 3.1 holds. Let $J = I : \operatorname{Im} Q \to \operatorname{Ker} L$, $(y_1, y_2) \to (y_1, y_2)^\top$, it follows that

$$\begin{aligned}
&\deg\{JQN(y_1, y_2)^\top, \Omega \cap \operatorname{Ker}L, (0,0)^\top\} \\
&= \operatorname{sgn}\left\{\left[\bar{a}_{11}\bar{b}_2 \sum_{p=0}^{M} k_1(p) - \bar{a}_{21}\bar{b}_1 \sum_{p=0}^{M} k_2(p)\right] e^{2y_1^* + y_2^*}\right. \\
&\quad \left. + [\bar{a}_{11}\bar{a}_{22} - \bar{a}_{12}\bar{a}_{21}]\, e^{y_1^* + y_2^*} + \left[\bar{a}_{22}\bar{b}_1 \sum_{p=0}^{M} k_2(p) - \bar{a}_{12}\bar{b}_2 \sum_{p=0}^{M} k_1(p)\right] e^{y_1^* + 2y_2^*}\right\} \\
&= 1 \neq 0.
\end{aligned}$$

Finally, we will show that $N$ is $L-$compact on $\bar{\Omega}$. For any $y \in \bar{\Omega}$, we have

$$\begin{aligned}
\|QNy\| = \left\|\frac{1}{\omega}\sum_{s=0}^{\omega-1} Ny(x)\right\| &\leq \max\left\{r_1^U + a_{11}^U e^{H_1} + a_{12}^U e^{H_2}\right. \\
&\quad + b_1^U (M+1) k_2^U e^{H_1 + H_2}, r_2^U + a_{21}^U e^{H_1} + a_{22}^U e^{H_2} \\
&\quad \left. + b_2^U (M+1) k_1^U e^{H_1 + H_2}\right\} \doteq E.
\end{aligned}$$

Hence, $QN(\bar{\Omega})$ is bounded. Obviously, $QNy : \bar{\Omega} \to Y$ is continuous.

It is easy to see that

$$\begin{aligned}
\|K_p(I-Q)Ny\| &\leq \sum_{s=0}^{\omega-1} \|Ny(s)\| + \frac{1}{\omega}\sum_{s=0}^{\omega-1}(\omega - s)\,\|Ny(s)\| \\
&\quad + \frac{1+3\omega}{2\omega}\sum_{s=0}^{\omega-1}\|Ny(s)\| \leq \frac{1+7\omega}{2}E.
\end{aligned}$$

For any $y \in \bar{\Omega}$, $n_1, n_2 \in I_\omega$, without loss of generality, let $n_2 > n_1$, we have

$$\begin{aligned}
|K_p(I-Q)Ny(n_2) - K_p(I-Q)Ny(n_1)| &= \left|\sum_{s=n_1}^{n_2-1} Ny(s) - \frac{n_2 - n_1}{\omega}\sum_{s=0}^{\omega-1} Ny(s)\right| \\
&\leq \sum_{s=n_1}^{n_2-1} |Ny(s)| + \frac{n_2 - n_1}{\omega}\sum_{s=0}^{\omega-1}|Ny(s)| \\
&\leq 2E|n_2 - n_1|.
\end{aligned}$$

Thus, the set $\{K_p(I-Q)Ny \big| y \in \bar{\Omega}\}$ is equicontinuous and uniformly bounded. By using the Arzela-Ascoli theorem, we see that $K_p(I-Q)N : \bar{\Omega} \to X$ is compact. Consequently, $N$ is $L-$compact.

By now we know that $\Omega$ verifies all the requirements in Lemma 3.1 and then system (3.3) has at least one $\omega-$periodic solution. By the medium of (3.2), we derive that (1.6) has at least one positive $\omega-$periodic solution. This completes the proof. □

Next, by constructing a Lyapunov function, we further research the global attractivity of positive periodic solutions of (1.6), sufficient conditions which guarantee the global attractivity of positive periodic solutions are obtained. We first give the definition of global attractivity.

**Definition 3.1** A positive periodic solution $\{(x_1^*(n), x_2^*(n))^\top\}$ of system (1.6) is said to be globally attractive if each other solution $\{(x_1(n), x_2(n))^\top\}$ of (1.6) satisfies

$$\lim_{n\to+\infty} |x_1(n) - x_1^*(n)| = 0, \quad \lim_{n\to+\infty} |x_2(n) - x_2^*(n)| = 0.$$

**Theorem 3.2** *In addition to assumption (3.1), assume further that there exists a positive constant $\eta$ such that*

$$\min\left\{a_{ii}^l, \frac{2}{M_i} - a_{ii}^u\right\} - a_{ji}^u - M_j(M+1)(b_i^u k_j^u + b_j^u k_i^u) \geq \eta \tag{3.14}$$

*for $i, j = 1, 2$ and $i \neq j$. Then the positive periodic solution of system (1.6) is globally attractive.*

*Proof* Let $\{(x_1^*(n), x_2^*(n))^\top\}$ be a positive periodic solution of (1.6), we prove below that it is globally attractive.

We first let $V_{11}(n) = \left|\ln x_1(n) - \ln x_1^*(n)\right|$, then it follows from the first equation of (1.6) that

$$\begin{aligned}
V_{11}(n+1) &= \left|\ln x_1(n+1) - \ln x_1^*(n+1)\right| \\
&= \Bigg|\Bigg[\ln x_1(n) + r_1(n) - a_{11}(n)x_1(n) - a_{12}(n)x_2(n) \\
&\qquad -b_1(n)x_1(n)\sum_{p=0}^{M} k_2(p)x_2(n-p)\Bigg] \\
&\qquad -\Bigg[\ln x_1^*(n) + r_1(n) - a_{11}(n)x_1^*(n) - a_{12}(n)x_2^*(n) \\
&\qquad -b_1(n)x_1^*(n)\sum_{p=0}^{M} k_2(p)x_2^*(n-p)\Bigg]\Bigg| \\
&\leq |\ln x_1(n) - \ln x_1^*(n) - a_{11}(n)[x_1(n) - x_1^*(n)]|
\end{aligned}$$

$$+a_{12}(n)\big|x_2(n)-x_2^*(n)\big|+b_1(n)\sum_{p=0}^{M}k_2(p)\big|x_1(n)x_2(n-p)$$
$$-x_1^*(n)x_2^*(n-p)\big|. \tag{3.15}$$

Applying the mean value theorem, we get

$$x_1(n)-x_1^*(n)=e^{\ln x_1(n)}-e^{\ln x_1^*(n)}=\theta_1(n)[\ln x_1(n)-\ln x_1^*(n)],$$

that is,

$$\ln x_1(n)-\ln x_1^*(n)=\frac{1}{\theta_1(n)}[x_1(n)-x_1^*(n)],$$

where $\theta_1(n)$ lies between $x_1(n)$ and $x_1^*(n)$. Then we have

$$\begin{aligned}
&\big|\ln x_1(n)-\ln x_1^*(n)-a_{11}(n)[x_1(n)-x_1^*(n)]\big|\\
&\quad=\big|\ln x_1(n)-\ln x_1^*(n)\big|-\big|\ln x_1(n)-\ln x_1^*(n)\big|\\
&\qquad+|\ln x_1(n)-\ln x_1^*(n)-a_{11}(n)[x_1(n)-x_1^*(n)]|\\
&\quad=\big|\ln x_1(n)-\ln x_1^*(n)\big|-\frac{1}{\theta_1(n)}\big|x_1(n)-x_1^*(n)\big|\\
&\qquad+|\frac{1}{\theta_1(n)}[x_1(n)-x_1^*(n)]-a_{11}(n)[x_1(n)-x_1^*(n)]|\\
&\quad=\big|\ln x_1(n)-\ln x_1^*(n)\big|-\frac{1}{\theta_1(n)}\big|x_1(n)-x_1^*(n)\big|+\Big|\frac{1}{\theta_1(n)}\\
&\qquad-a_{11}(n)\Big|\times\big|x_1(n)-x_1^*(n)\big|\\
&\quad=\big|\ln x_1(n)-\ln x_1^*(n)\big|-\Big(\frac{1}{\theta_1(n)}-\Big|\frac{1}{\theta_1(n)}\\
&\qquad-a_{11}(n)\Big|\Big)\times\big|x_1(n)-x_1^*(n)\big|.
\end{aligned}\tag{3.16}$$

And hence it follows from (3.15) and (3.16) that

$$\begin{aligned}
\Delta V_{11}(n)&=V_{11}(n+1)-V_{11}(n)\\
&\le-\Big(\frac{1}{\theta_1(n)}-\Big|\frac{1}{\theta_1(n)}-a_{11}(n)\Big|\Big)\times\big|x_1(n)-x_1^*(n)\big|\\
&\qquad+a_{12}(n)\big|x_2(n)-x_2^*(n)\big|+b_1(n)\sum_{p=0}^{M}k_2(p)|x_1(n)x_2(n-p)\\
&\qquad-x_1^*(n)x_2^*(n-p)\big|.
\end{aligned}\tag{3.17}$$

According to (2.3) and (2.4), for any positive constant $\varepsilon$, there exists an $n_0\in\mathbb{Z}^+$ such that $x_1(n)\le M_1+\varepsilon,\ \ x_2(n)\le M_2+\varepsilon$ for $n\ge n_0$. Therefore, for all $n\ge n_0+M$, $p=0,1,2,\ldots,M$, we can obtain that

$$
\begin{aligned}
&|x_1(n)x_2(n-p) - x_1^*(n)x_2^*(n-p)| \\
&\quad = |x_1(n)x_2(n-p) - x_1(n)x_2^*(n-p) + x_1(n)x_2^*(n-p) - x_1^*(n)x_2^*(n-p)| \\
&\quad = |x_1(n)[x_2(n-p) - x_2^*(n-p)] + x_2^*(n-p)[x_1(n) - x_1^*(n)]| \\
&\quad \le (M_1+\varepsilon)|x_2(n-p) - x_2^*(n-p)| + (M_2+\varepsilon)|x_1(n) - x_1^*(n)|.
\end{aligned}
$$

Then, for all $n \ge n_0 + M$, (3.17) can be rewritten as

$$
\begin{aligned}
\Delta V_{11}(n) = V_{11}(n+1) - V_{11}(n) &\le -\Big(\frac{1}{\theta_1(n)} - \Big|\frac{1}{\theta_1(n)} - a_{11}(n)\Big| \\
&\quad -b_1(n)(M_2+\varepsilon)\sum_{p=0}^{M} k_2(p)\Big) \times \big|x_1(n) - x_1^*(n)\big| \\
&\quad +a_{12}(n)\big|x_2(n) - x_2^*(n)\big| \\
&\quad +b_1(n)(M_1+\varepsilon)\sum_{p=0}^{M} k_2(p)\big|x_2(n-p) - x_2^*(n-p)\big|. \qquad (3.18)
\end{aligned}
$$

Next, we let

$$
V_{12}(n) = \sum_{p=0}^{M}\sum_{s=n-p}^{n-1} b_1(s+p)(M_1+\varepsilon)k_2(p)\big|x_2(s) - x_2^*(s)\big|.
$$

By a simple calculation, it derives that

$$
\begin{aligned}
\Delta V_{12}(n) &= V_{12}(n+1) - V_{12}(n) \\
&= \sum_{p=0}^{M}\sum_{s=n+1-p}^{n} b_1(s+p)(M_1+\varepsilon)k_2(p)\big|x_2(s) - x_2^*(s)\big| \\
&\quad -\sum_{p=0}^{M}\sum_{s=n-p}^{n-1} b_1(s+p)(M_1+\varepsilon)k_2(p)\big|x_2(s) - x_2^*(s)\big| \\
&= \sum_{p=0}^{M}\sum_{s=n+1-p}^{n-1} b_1(s+p)(M_1+\varepsilon)k_2(p)\big|x_2(s) - x_2^*(s)\big| \\
&\quad +\sum_{p=0}^{M} b_1(n+p)(M_1+\varepsilon)k_2(p)\big|x_2(n) - x_2^*(n)\big| \\
&\quad -\sum_{p=0}^{M}\sum_{s=n+1-p}^{n-1} b_1(s+p)(M_1+\varepsilon)k_2(p)\big|x_2(s) - x_2^*(s)\big|
\end{aligned}
$$

$$
\begin{aligned}
&\quad -\sum_{p=0}^{M} b_1(n)(M_1+\varepsilon)k_2(p)\big|x_2(n-p)-x_2^*(n-p)\big| \\
&= \sum_{p=0}^{M} b_1(n+p)(M_1+\varepsilon)k_2(p)\big|x_2(n)-x_2^*(n)\big| \\
&\quad -\sum_{p=0}^{M} b_1(n)(M_1+\varepsilon)k_2(p)\big|x_2(n-p)-x_2^*(n-p)\big|. \qquad (3.19)
\end{aligned}
$$

We can define $V_1(n) = V_{11}(n) + V_{12}(n)$, thus it follows from (3.18) and (3.19) for all $n \geq n_0 + M$ that

$$
\begin{aligned}
\Delta V_1(n) &= \Delta V_{11}(n) + \Delta V_{12}(n) \\
&\leq -\left(\frac{1}{\theta_1(n)} - \left|\frac{1}{\theta_1(n)} - a_{11}(n)\right| - b_1(n)(M_2+\varepsilon)\sum_{p=0}^{M} k_2(p)\right) \times \big|x_1(n)-x_1^*(n)\big| \\
&\quad + \left[a_{12}(n) + \sum_{p=0}^{M} b_1(n+p)(M_1+\varepsilon)k_2(p)\right] \times \big|x_2(n)-x_2^*(n)\big|. \qquad (3.20)
\end{aligned}
$$

Similar to above argument, we can define $V_2(n) = V_{21}(n) + V_{22}(n)$, where

$$
\begin{aligned}
V_{21}(n) &= \big|\ln x_2(n) - \ln x_2^*(n)\big|, \\
V_{22}(n) &= \sum_{p=0}^{M} \sum_{s=n-p}^{n-1} b_2(s+p)(M_2+\varepsilon)k_1(p)\big|x_1(s) - x_1^*(s)\big|.
\end{aligned}
$$

Then for all $n \geq n_0 + M$, it is easy to obtain that

$$
\begin{aligned}
\Delta V_2(n) &= \Delta V_{21}(n) + \Delta V_{22}(n) \\
&\leq -\left(\frac{1}{\theta_2(n)} - \left|\frac{1}{\theta_2(n)} - a_{22}(n)\right| - b_2(n)(M_1+\varepsilon)\sum_{p=0}^{M} k_1(p)\right) \times \big|x_2(n)-x_2^*(n)\big| \\
&\quad + \left[a_{21}(n) + \sum_{p=0}^{M} b_2(n+p)(M_2+\varepsilon)k_1(p)\right] \times \big|x_1(n)-x_1^*(n)\big|, \qquad (3.21)
\end{aligned}
$$

where $\theta_2(n)$ lies between $x_2(n)$ and $x_2^*(n)$.

Now, we are in a position to define $V(n) = V_1(n) + V_2(n)$. Obviously, $V(n) \geq 0$ for all $n \in \mathbb{Z}$ and $V(n_0 + M) < +\infty$. For the arbitrariness of $\varepsilon$, in view of (3.14), we can choose a small enough $\varepsilon > 0$ such that

$$
\min\left\{a_{ii}^l, \frac{2}{M_i+\varepsilon} - a_{ii}^u\right\} - a_{ji}^u - (M_j+\varepsilon)(M+1)(b_i^u k_j^u + b_j^u k_i^u) \geq \eta \quad (3.22)
$$

for $i, j = 1, 2$ and $i \neq j$. Therefore, combining (3.20) and (3.21), for all $n \geq n_0 + M$, we have

$$\begin{aligned}
\Delta V(n) = \Delta V_1(n) + \Delta V_2(n) \leq -\sum_{i=1}^{2} \Bigg\{ \frac{1}{\theta_i(n)} - \left| \frac{1}{\theta_i(n)} - a_{ii}(n) \right| \\
-a_{ji}(n) - b_i(n)(M_j + \varepsilon) \sum_{p=0}^{M} k_j(p) - \sum_{p=0}^{M} b_j(n+p) k_i(p)(M_j + \varepsilon) \Bigg\} \\
\times \left| x_i(n) - x_i^*(n) \right| \leq -\sum_{i=1}^{2} \Big\{ \min\Big\{ a_{ii}^l, \frac{2}{M_i + \varepsilon} - a_{ii}^u \Big\} - a_{ji}^u \\
-(M_j + \varepsilon)(M+1)(b_i^u k_j^u + b_j^u k_i^u) \Big\} \left| x_i(n) - x_i^*(n) \right| \\
\leq -\eta \sum_{i=1}^{2} \left| x_i(n) - x_i^*(n) \right|, \qquad i, j = 1, 2 \text{ and } i \neq j. \qquad (3.23)
\end{aligned}$$

Summing both sides of (3.23) from $n_0 + M$ to $n$, it derives that

$$\sum_{k=n_0+M}^{n} \Delta V(k) = \sum_{k=n_0+M}^{n} [V(k+1) - V(k)] \leq -\eta \sum_{k=n_0+M}^{n} \sum_{i=1}^{2} \left| x_i(k) - x_i^*(k) \right|,$$

which implies,

$$V(n+1) + \eta \sum_{k=n_0+M}^{n} \sum_{i=1}^{2} \left| x_i(k) - x_i^*(k) \right| \leq V(0).$$

It follows that

$$\sum_{k=n_0+M}^{n} \sum_{i=1}^{2} \left| x_i(k) - x_i^*(k) \right| \leq \frac{V(0)}{\eta}.$$

By the fundamental theorem of positive series, we can derive that

$$\sum_{k=n_0+M}^{\infty} \sum_{i=1}^{2} \left| x_i(k) - x_i^*(k) \right| \leq \frac{V(0)}{\eta} < +\infty,$$

that is,

$$\lim_{n \to +\infty} \sum_{i=1}^{2} \left| x_i(n) - x_i^*(n) \right| = 0,$$

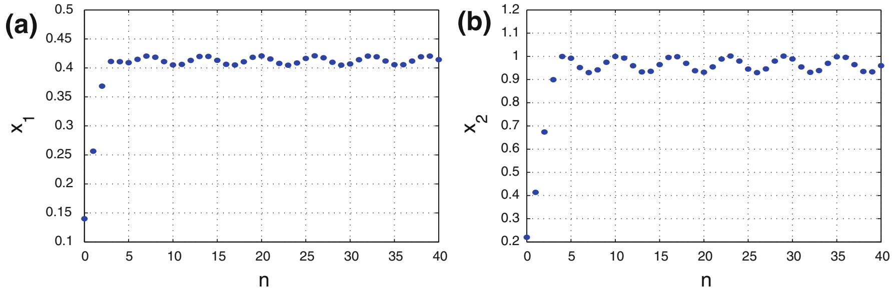


**Fig. 1** Permanence of system (4.1) with $\{(x_1(-1), x_2(-1))^\top\} = \{(0.35, 0.32)^\top\}$ and $\{(x_1(0), x_2(0))^\top\} = \{(0.14, 0.22)^\top\}$. **a** Time-series of $x_1$. **b** Time-series of $x_2$

and we can easily see that

$$\lim_{n\to+\infty} \left|x_1(n) - x_1^*(n)\right| = 0, \quad \lim_{n\to+\infty} \left|x_2(n) - x_2^*(n)\right| = 0,$$

which implies $\{(x_1^*(n), x_2^*(n))^\top\}$ is globally attractive. This completes the proof. □

## 4 Example and numerical simulation

In this paper, we investigate the dynamic behaviors of a non-autonomous two species delay difference system of plankton allelopathy, such as permanence, existence and global attractivity of positive periodic solutions. In the following, we will give two examples to illustrate the feasibility of our results in Theorems 2.1, 3.1 and 3.2.

*Example 4.1* Consider the following system

$$\begin{cases} x_1(n+1) = x_1(n)\exp\{0.85+0.04\sin n - (1.83+0.02\sin n)x_1(n) \\ \quad -(0.03+0.01\sin n)x_2(n) - (0.1+0.01\sin n)x_1(n)\,[0.83x_2(n)+0.83x_2(n-1)]\}, \\ x_2(n+1) = x_2(n)\exp\{0.80+0.01\sin n - (0.02+0.01\sin n)x_1(n) \\ \quad -(0.70+0.03\sin n)x_2(n) - (0.2+0.01\sin n)x_2(n)\,[0.73x_1(n)+0.73x_1(n-1)]\}. \end{cases} \tag{4.1}$$

By a simple calculation, we have

$$r_1^l - a_{12}^u M_2 \approx 0.7606 > 0, \quad r_2^l - a_{21}^u M_1 \approx 0.7752 > 0,$$
$$\Delta_1 M_1 \approx 1.3504 > 1, \quad \Delta_2 M_2 \approx 1.4040 > 1.$$

According to Theorem 2.1, system (4.1) is permanent (see Fig. 1).

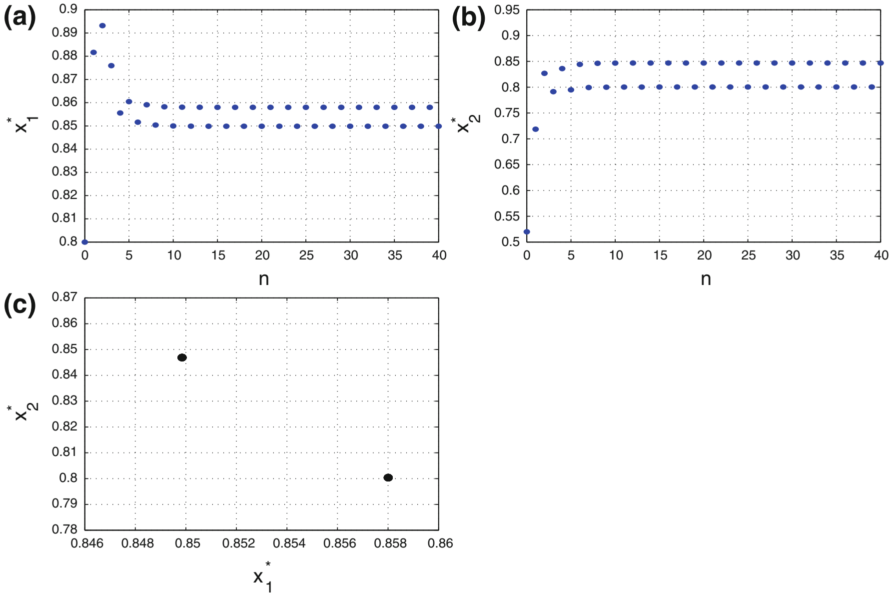


**Fig. 2** Dynamic behaviors of system (4.2) with $\{(x_1^*(-1), x_2^*(-1))^\top\} = \{(0.52, 0.63)^\top\}$ and $\{(x_1^*(0), x_2^*(0))^\top\} = \{(0.80, 0.52)^\top\}$. **a** Time-series of $x_1^*$. **b** Time-series of $x_2^*$. **c** Phase portrait of $x_1^*$ and $x_2^*$ with $n$ over [20, 40]

*Example 4.2* Consider the following periodic system

$$\begin{cases} x_1(n+1) = x_1(n)\exp\{0.85+0.04\cos\pi n-(0.83+0.02\cos\pi n)x_1(n) \\ -(0.03+0.01\cos\pi n)x_2(n)-(0.1+0.01\cos\pi n)x_1(n)\,[0.83x_2(n)+0.83x_2(n-1)]\}, \\ x_2(n+1) = x_2(n)\exp\{0.80+0.01\cos\pi n-(0.02+0.01\cos\pi n)x_1(n) \\ -(0.70+0.03\cos\pi n)x_2(n)-(0.2+0.01\cos\pi n)x_2(n)\,[0.73x_1(n)+0.73x_1(n-1)]\}. \end{cases} \tag{4.2}$$

Clearly,

$$\frac{\bar{a}_{11}}{\bar{a}_{21}} \approx 42.5000 \ge \min\left\{\frac{\bar{b}_1\sum_{p=0}^{M}k_2(p)}{\bar{b}_2\sum_{p=0}^{M}k_1(p)}, \frac{\bar{r}_1}{\bar{r}_2}e^{2\bar{r}_1\omega}\right\} \approx 0.5685,$$

$$\frac{\bar{a}_{22}}{\bar{a}_{12}} \approx 23.3333 \ge \min\left\{\frac{\bar{b}_2\sum_{p=0}^{M}k_1(p)}{\bar{b}_1\sum_{p=0}^{M}k_2(p)}, \frac{\bar{r}_2}{\bar{r}_1}e^{2\bar{r}_2\omega}\right\} \approx 1.7590,$$

$$\min\left\{a_{11}^l, \frac{2}{M_1} - a_{11}^u\right\} - a_{21}^u - M_2(M+1)(b_1^u k_2^u + b_2^u k_1^u) \approx 0.1366 > 0,$$

$$\min\left\{a_{22}^l, \frac{2}{M_2} - a_{22}^u\right\} - a_{12}^u - M_1(M+1)(b_2^u k_1^u + b_1^u k_2^u) \approx 0.0890 > 0.$$

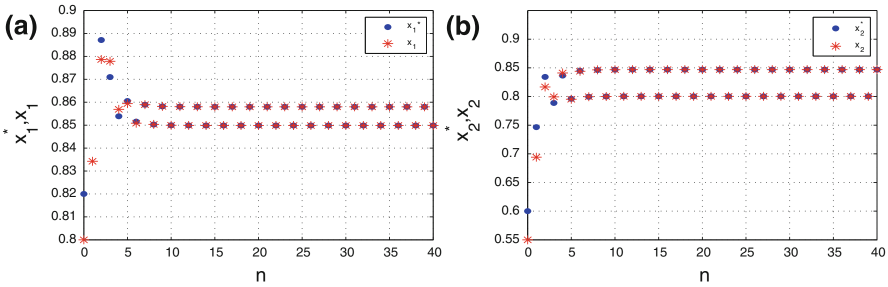

**Fig. 3** Dynamic behaviors of system (4.2) with different initial values. **a** Time-series of $x_1^*$ with $x_1^*(-1) = 0.7, x_1^*(0) = 0.82$ and $x_1$ with $x_1(-1) = 0.9, x_1(0) = 0.8$. **b** Time-series of $x_2^*$ with $x_2^*(-1) = 0.5, x_2^*(0) = 0.6$ and $x_2$ with $x_2(-1) = 1.0, x_2(0) = 0.55$

According to Theorems 3.1 and 3.2, system (4.2) has one positive 2-periodic solution $\{(x_1^*(n), x_2^*(n))^\top\}$ (see Fig. 2), which is globally attractive (see Fig. 3). That is, any positive solution $\{(x_1(n), x_2(n))^\top\}$ tends to $\{(x_1^*(n), x_2^*(n))^\top\}$. From Fig. 3(a) and Fig. 3(b), we can see that $x_1(n)$ and $x_2(n)$ tend to $x_1^*(n)$ and $x_2^*(n)$, respectively.

**Acknowledgments** The work is supported by the Key Project of Chinese Ministry of Education (No.210134), the Innovation Term of Educational Department of Hubei Province of China (No.T200804), the Innovation Term of Hubei University for Nationalities (No.MY2009T001), the Scientific Research Foundation of the Education Department of Hubei Province, China(No.D20101902), and the Foundation of Cultivating Excellent Master Dissertation of Hubei Unversity for Nationalities.

## References

1. M. Hans, *Der einfluss einer pflanze auf die andere: allelopathie* (Verlag von Gustav Fisher, Jena, 1937)
2. S.A. Huntsman, R.T. Barber, Modification of phytoplankton growth by excreted compounds in low density populations. J. Phycol. **11**, 10–13 (1975)
3. E.L. Rice, *Allelopathy*, 2nd edn. (Academic Press, New York, 1984)
4. J.M. Wolf, E.L. Rice, Allelopathic interaction between algae. J. Chem. Ecol. **5**, 533–542 (1979)
5. Z.J. Liu, L.S. Chen, Positive periodic solution of a general discrete non-autonomous difference system of plankton allelopathy with delays. J. Comput. Appl. Math. **197**, 446–456 (2006)
6. Z.J. Liu, L.S. Chen, Periodic solution of a two-species competitive system with toxicant and birth pulse. Chaos Soliton Fract. **32**, 1703–1712 (2007)
7. Z.J. Liu, J.H. Wu, Y.P. Chen, M. Haque, Impulsive perturbations in a periodic delay differential equation model of plankton allelopathy. Nonlinear Anal. RWA. **11**, 432–445 (2010)
8. Z.J. Liu, J. Hui, J.H. Wu, Permanence and partial extinction in an impulsive delay competitive system with the effect of toxic substances. J. Math. Chem. **46**, 1213–1231 (2009)
9. C.R. Tian, Z.G. Lin, Asymptotic behavior of solutions of a periodic diffusion system of plankton allelopathy. Nonlinear Anal. RWA. **3**, 1581–1588 (2010)
10. C.R. Tian, L. Zhang, Z. Ling, The stability of a diffusion model of plankton allelopathy with spatio Ctemporal delays. Nonlinear Anal. RWA. **10**, 2036–2046 (2009)
11. F.D. Chen, Z. Li, X.X. Chen, J. Laitochova, Dynamic behaviors of a delay differential equation model of plankton allelopathy. J. Comput. Appl. Math. **206**, 733–754 (2007)
12. H.F. Huo, W.T. Li, Permanence and global stability for nonautonomous discrete model of plankton allelopathy. Appl. Math. Lett. **17**, 1007–1013 (2004)
13. Y.P. Chen, F.D. Chen, Z. Li, Dynamics behaviors of a general discrete nonautonomous system of plankton alleopathy with delays Discrete Dynamics in Nature and Society Article ID 310425, (2008) p. 22
14. J.M. Smith, *Models in Ecology* (Cambridge University, Cambridge, 1974)

15. J. Chattopadhyay, Effect of toxic substances on a two species competition system. Ecol. Model. **84**, 287–289 (1996)
16. Z. Jin, Z.E. Ma, Periodic solutions for delay differential equations model of plankton allelopathy. Comput. Math. Appl. **44**, 491–500 (2002)
17. R.P. Agarwal, *Difference Equations and Inequalities: Theory, Methods and Applications* (Marcel dekker, New York, 2000)
18. H.I. Freedman, *Deterninistic Mathematical Models in population Ecology* (Marcel Dekker, New York, 1980)
19. J.D. Murry, *Mathematical Biology* (Springer, Berlin, 1989)
20. K. Gopalsamy, I. Leung, Delay induced periodicity in a neural netlet of excitation and inhibition. Phsica D **89**, 395–426 (1996)
21. P. Liu, X. Cui, A discrete model of competition. Math. Comput. Simul. **49**, 1–12 (1999)
22. S. Mohamad, K. Golpalsamy, Dynamics of a class of discrete-time neural networks and their continuous-time counterparts. Math. Comput. Simul. **53**, 1–39 (2000)
23. X.T. Yang, Uniform persistence and periodic solutions for a discrete predator-prey system with delays. J. Math. Anal. Appl. **316**, 161–177 (2006)
24. R. Gaines, J. Mawhin, *Coincidence Degree and Nonlinear Differential Equations* (Spring, Berlin, 1977)
25. M. Fan, K. Wang, Periodic solutions of a discrete time nonautonomous ratio dependent predator-prey system. Math. Comput. Model. **35**, 951–961 (2002)